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Difference-elliptic operators and root systems

By Ivan Cherednik*

Recently a new technique in the harmonic analysis on symmetric spaces was suggested based on certain remarkable representations of affine and double affine Hecke algebras in terms of Dunkl and Demazure operators instead of Lie groups and Lie algebras. In the classic case (see [O,H,C5]) it resulted (among other applications) in a new theory of radial part of Laplace operators and their deformations including a related concept of the Fourier transform (see [DJO]). Some observations indicate that there can be connections with the so-called W_∞-algebras.

In papers [C1,C2,C4] the analogous difference methods were developed to generalize the operators constructed by Macdonald (corresponding to the minuscule and certain similar weights) and those considered in [N] and other works on q-symmetric spaces. It is quite likely that the Fourier transform is self-dual in the difference setting (in contrast to the classical theory).

Paper [C3] is devoted to the differential-elliptic case presumably corresponding to the Kac-Moody algebras. Presumably because the ways of extending the traditional harmonic analysis to these algebras are still rather obscure although there are interesting projects. In the present paper we demonstrate that the new technique works well even in the most general difference-elliptic case conjecturally corresponding to the q-Kac-Moody algebras considered at the critical level.

We discuss here only the construction of the generalized radial (zonal) Laplace operators. These operators are closely related to the so-called quantum many-body problem (Calogero, Sutherland, Moser, Olshanetsky, Perelomov), the conformal field theory (Knizhnik- Zamolodchikov equations), and the classic theory of the hypergeometric functions. They are expected to have applications to the characters of the Kac-Moody algebras and in Arithmetic. The natural problem is to extend the Macdonald theory [M1,M2,C2] to the elliptic case.

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We also connect our operators with the difference-elliptic Ruijsenaars operators from [R] (corresponding to the minuscule weights of type A) generalizing in its turn the Olshanetsky-Perelomov differential elliptic operators.

At the end of the paper we show that the monodromy of the trigonometric KZ equation from [C4] leads (when properly defined) to a new invariant R-matrix of elliptic type (which might be of a certain importance to establish the relation between double affine Hecke algebras and proper "elliptic" quantum groups). The invariance means that the monodromy elements for W-conjugated simple reflections are conjugated with respect to the action of the Weyl group W. We note that the monodromy elements (matrices) always satisfy the braid relations. However if one defines the monodromy representation following [FR] and other papers devoted to trigonometric difference KZ equations then it is never invariant. The monodromy matrices for simple reflections are connected in a more complicated way (the relations are similar to the so-called Star-Triangle identities).

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1. Affine root systems

Let $R = \{\alpha\} \subset \mathbf{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a euclidean form (z, z') on $\mathbf{R}^n \ni z, z'$. The latter is unique up to proportionality. We assume that $(\alpha, \alpha) = 2$ for long α. Let us fix the set R_+ of positive roots ($R_- = -R_+$), the corresponding simple roots $\alpha_1, ..., \alpha_n$, and their dual counterparts $a_1, ..., a_n, a_i = \alpha_i^\vee$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. The dual fundamental weights $b_1, ..., b_n$ are determined from the relations $(b_i, a_j) = \delta_i^j$ for the Kronecker delta. We will also introduce the lattices

$$A = \oplus_{i=1}^n \mathbf{Z} a_i \subset B = \oplus_{i=1}^n \mathbf{Z} b_i,$$

and $A_\pm, B_\pm$ for $\mathbf{Z}_\pm = \{m \in \mathbf{Z}, \pm m \geq 0\}$ instead of $\mathbf{Z}$. (In the standard notations, $A = Q^\vee,\ B = P^\vee$.) Later on,

$$\begin{aligned} \nu_\alpha \ &= \ (\alpha, \alpha),\ \nu_i \ = \ \nu_{\alpha_i},\ \nu_R \ = \{\nu_\alpha, \alpha \in R\}, \\ \rho_\nu \ &= \ (1/2) \sum_{\nu_\alpha = \nu} \alpha \ = \ (\nu/2) \sum_{\nu_i = \nu} b_i, \quad \text{for} \quad \alpha \in R_+. \end{aligned} \tag{1.1}$$

The vectors $\tilde\alpha = [\alpha, k] \in \mathbf{R}^n \times \mathbf{R} \subset \mathbf{R}^{n+1}$ for $\alpha \in R, k \in \mathbf{Z}$ form the *affine root system* $R^a \supset R$ ($z \in \mathbf{R}^n$ are identified with $[z, 0]$). We add $\alpha_0 \overset{def}{=} [-\theta, 1]$ to the simple roots for the *maximal root* $\theta \in R$. The corresponding set R_+^a of positive roots coincides with $R_+ \cup \{[\alpha, k],\ \alpha \in R,\ k > 0\}$.

We denote the Dynkin diagram and its affine completion with $\{\alpha_j, 0 \le j \le n\}$ as the vertices by Γ and Γ^a. Let $m_{ij} = 2, 3, 4, 6$ if α_i and α_j are joined by 0,1,2,3 laces respectively. The set of the indices of the images of α_0 by all the automorphisms of Γ^a will be denoted by O ($O = \{0\}$ for E_8, F_4, G_2). Let $O^* = r \in O, r \neq 0$. The elements b_r for $r \in O^*$ are the so-called minuscule weights ($(b_r, \alpha) \le 1$ for $\alpha \in R_+$).

Given $\tilde\alpha = [\alpha, k] \in R^a$, $b \in B$, let

$$s_{\tilde\alpha}(\tilde z) \;=\; \tilde z - (z, \alpha^\vee)\tilde\alpha, \quad b'(\tilde z) \;=\; [z, \zeta - (z, b)] \tag{1.2}$$

for $\tilde z = [z, \zeta] \in \mathbf{R}^{n+1}$.

The *affine Weyl group* W^a is generated by all $s_{\tilde\alpha}$ (we write $W^a =< s_{\tilde\alpha}, \tilde\alpha \in R^a_+ >$). One can take the simple reflections $s_j = s_{\alpha_j}, 0 \le j \le n$, as its generators and introduce the corresponding notion of the length. This group is the semi-direct product $W \ltimes A'$ of its subgroups $W =< s_\alpha, \alpha \in R_+ >$ and $A' = \{a', a \in A\}$, where

$$a' = \; s_\alpha s_{[\alpha,1]} = \; s_{[-\alpha,1]} s_\alpha \;\text{ for }\; a = \alpha^\vee, \;\alpha \in R. \tag{1.3}$$

The *extended Weyl group* W^b generated by W and B' (instead of A') is isomorphic to $W \ltimes B'$:

$$(wb')([z, \zeta]) \;=\; [w(z), \zeta - (z, b)] \;\text{ for }\; w \in W, b \in B. \tag{1.4}$$

Definition 1.1. *i) Given $b_+ \in B_+$, let*

$$\omega_{b_+} = w_0 w_0^+ \in W, \;\pi_{b_+} = b'_+(\omega_{b_+})^{-1} \;\in\; W^b, \;\omega_i = \omega_{b_i}, \pi_i = \pi_{b_i}, \tag{1.5}$$

where w_0 (respectively, w_0^+) is the longest element in W (respectively, in W_{b_+} generated by s_i preserving b_+) relative to the set of generators $\{s_i\}$ for $i > 0$.

ii) If b is arbitrary then there exist unique elements $w \in W$, $b_+ \in B_+$ such that $b = w(b_+)$ and $(\alpha, b_+) \neq 0$ if $(-\alpha) \in R_+ \ni w(\alpha)$. We set

$$b' = \pi_b \omega_b, \quad \textit{where} \;\; \omega_b = \omega_{b_+} w^{-1}, \pi_b = w\pi_{b_+}. \tag{1.6}$$

□

We will mostly use the elements $\pi_r = \pi_{b_r}, r \in O$. They leave Γ^a invariant and form a group denoted by Π, which is isomorphic to B/A by the natural projection $\{b_r \to \pi_r\}$. As to $\{\omega_r\}$, they preserve the set $\{-\theta, \alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (\omega_r)^{-1}(-\theta)$ distinguish the indices $r \in O^*$. These elements are important because due to [B,V,C4]:

$$W^b = \Pi \ltimes W^a, \;\text{ where }\; \pi_r s_i \pi_r^{-1} = s_j \;\text{ if }\; \pi_r(\alpha_i) = \alpha_j, \;0 \le j \le n. \tag{1.7}$$

We extend the notion of the length to W^b. Given $\nu \in \nu_R$, $r \in O^*$, $\tilde w \in W^a$, and a reduced decomposition $\tilde w \;=\; s_{j_l}...s_{j_2}s_{j_1}$ with respect to $\{s_j, 0 \le j \le n\}$,

we call $l = l(\hat{w})$ the *length* of $\hat{w} = \pi_r\tilde{w} \in W^b$. Setting

$$\begin{aligned}\lambda(\hat{w}) =&\{\tilde{\alpha}^1 = \alpha_{j_1},\ \tilde{\alpha}^2 = s_{j_1}(\alpha_{j_2}),\ \tilde{\alpha}^3 = s_{j_1}s_{j_2}(\alpha_{j_3}), \ldots \\ &\ldots, \tilde{\alpha}^l = \tilde{w}^{-1}s_{j_l}(\alpha_{j_l})\},\end{aligned} \tag{1.8}$$

we introduce the *partial lengths*:

$$l = \sum_\nu l_\nu,\ l_\nu = l_\nu(\hat{w}) = |\lambda_\nu(\hat{w})|,$$

where $|\ |$ denotes the number of elements,

$$\lambda_\nu(\hat{w}) = \{\tilde{\alpha}^m,\ \nu(\tilde{\alpha}^m) = \nu(\tilde{\alpha}_{j_m}) = \nu\}, 1 \le m \le l, \tag{1.9}$$

for $\nu([\alpha, k]) \stackrel{def}{=} \nu_\alpha$.

To check that these sets do not depend on the choice of the reduced decomposition let us use the affine action of W^b on $z \in \mathbf{R}^n$:

$$\begin{aligned}(wb')\langle z\rangle &= w(b + z),\ w \in W, b \in B, \\ s_{\tilde{\alpha}}\langle z\rangle &= z - ((z, \alpha) + k)\alpha^\vee,\ \tilde{\alpha} = [\alpha, k] \in R^a.\end{aligned} \tag{1.10}$$

and the affine Weyl chamber:

$$C^a = \bigcap_{j=0}^n L_{\alpha_j},\ L_{\tilde{\alpha}} = \{z \in \mathbf{R}^n,\ (z, \alpha) + k > 0\}.$$

PROPOSITION 1.2.

$$\begin{aligned}\lambda_\nu(\hat{w}) &= \{\tilde{\alpha} \in R^a,\ \hat{w}^{-1}\langle C^a\rangle \not\subset L_{\tilde{\alpha}},\ \nu(\tilde{\alpha}) = \nu\} \\ &= \{\tilde{\alpha} \in R^a,\ l_\nu(\hat{w}s_{\tilde{\alpha}}) < l_\nu(\hat{w})\}.\end{aligned} \tag{1.11}$$

□

We mention that

$$\begin{aligned}&l(\ \hat{w}s_{\tilde{\alpha}\{1\}}...s_{\tilde{\alpha}\{p\}}\ ) > l(\ \hat{w}s_{\tilde{\alpha}\{1\}}...s_{\tilde{\alpha}\{p+1\}}\ ), \text{ if} \\ &\tilde{\alpha}\{q\} \stackrel{def}{=} \tilde{\alpha}^{m_q},\ l \ge m_1 > m_2 > ... > m_p > m_{p+1} \ge 1.\end{aligned} \tag{1.12}$$

PROPOSITION 1.3. *Each of the following conditions for $x, y \in W^b$ is equivalent to the relation $l_\nu(xy) = l_\nu(x) + l_\nu(y)$:*

$$\begin{aligned}&a)\ \lambda_\nu(xy) = \lambda_\nu(y) \cup y^{-1}(\lambda_\nu(x)),\ b)\ y^{-1}(\lambda_\nu(x)) \subset R^a_+ \\ &c)\ \lambda_\nu(y) \subset \lambda_\nu(xy), \quad d) y^{-1}(\lambda_\nu(x)) \subset \lambda_\nu(xy).\end{aligned} \tag{1.13}$$

□

PROPOSITION 1.4. *i) In the above notations,*

$$\begin{aligned}&\lambda(b') = \{\tilde{\alpha}, \alpha \in R_+, (b, \alpha) > k \ge 0\} \cup \{\tilde{\alpha}, \alpha \in R_-, (b, \alpha) \ge k > 0\}, \\ &\lambda(\pi_b^{-1}) = \{\tilde{\alpha}, -(b, \alpha) > k \ge 0\}, \quad \textit{where}\ \ \tilde{\alpha} = [\alpha, k] \in R^a_+, b \in B.\end{aligned} \tag{1.14}$$

ii) If $\hat w \in b'W$ (i.e. $\hat w\langle 0\rangle = b$) then $\hat w = \pi_b w$ for $w \in W$ such that $l(\hat w) = l(\pi_b) + l(w)$. Given $b \in B$, this property (valid for any $\hat w$ taking 0 to b) determines π_b uniquely.

□

Relation (1.14) gives the following useful formulas:

$$
\begin{aligned}
l_\nu(b') &= \sum_\alpha |(b,\alpha)|, \alpha \in R_+, \nu_\alpha = \nu \in \nu_R,\\
l_\nu(b'_+) &= 2(b,\rho_\nu), \quad \text{when} \quad b \in B_+.
\end{aligned} \tag{1.15}
$$

Here $|\ |$ = absolute value. Moreover

$$
\lambda(\omega_{b'_+}) = \{\alpha \in R_+, (b_+,\alpha) > 0\} \quad \text{for} \quad b_+ \in B_+. \tag{1.16}
$$

We set

$$
c \preceq b, b \succeq c \quad \text{for} \quad b, c \in B \quad \text{if} \quad b - c \in A_+. \tag{1.17}
$$

Given $b \in B$, let $b_+ = w_+^{-1}(b) \in B_+$ for w_+ from Definition 1.1. The set

$$
\sigma^\vee(b) \stackrel{def}{=} \{c \in B, w(c) \preceq b_+ \quad \text{for} \quad \text{any } w \in W\} \tag{1.18}
$$

is W-invariant (which is evident) and convex. By *convex*, we mean that if $c, c^* = c + r\alpha^\vee \in \sigma^\vee(b)$ for $\alpha \in R, r \in \mathbf{Z}_+$, then

$$
\{c,\ c+\alpha^\vee, ..., c+(r-1)\alpha^\vee,\ c^*\} \subset \sigma^\vee(b). \tag{1.19}
$$

We also note that $\sigma^\vee(b)$ contains the orbit $W(b)$.

Proposition 1.5. *Given $\hat w \in W^b, \tilde\alpha \in \lambda(\hat w^{-1})$, let $b = \hat w\langle 0\rangle, \hat w_* = s_{\tilde\alpha}\hat w, b_* = \hat w_*\langle 0\rangle$. Then $b_* \in \sigma^\vee(b)$,*

$$
\begin{aligned}
&\ell(\hat w_*) < \ell(\hat w) \quad \textit{if} \quad b_* \neq b, \quad \textit{where} \quad \ell(\hat w) = \ell(b') \stackrel{def}{=} l(\pi_b),\\
&\ell(b'_+) < \ell(\hat w) < l(b') = \ell(b'_-) \quad \textit{if} \quad b_+ \neq b \neq b_- \stackrel{def}{=} -b_+.
\end{aligned} \tag{1.20}
$$

Moreover $\ell(\hat w_*) < \ell(\hat w)$ for $\hat w_* = s_{\tilde\alpha\{p\}}...s_{\tilde\alpha\{1\}}\hat w$, where we take any sequence (1.12) for $\hat w^{-1}$ (instead of $\hat w$) such that $\ell(s_{\tilde\alpha\{1\}}\hat w) < \ell(\hat w)$.

□

2. Affine R-matrices

We fix an arbitrary $\mathbf{C}$-algebra $\mathfrak{F}$. We introduce $\mathfrak{F}$-valued (abstract) affine R-matrices following [C4] (see also [C3]), define the monodromy cocycle of the corresponding Knizhnik-Zamolodchikov equation, and extend the latter to a new affine R-matrix. Starting with the basic trigonometric ones from [C4], this

construction gives R-matrices of elliptic type. We will use the notations from Section 1. Let us denote $\mathbf{R}\tilde\alpha+\mathbf{R}\tilde\beta\subset\mathbf{R}^n$ by $\mathbf{R}\langle\tilde\alpha,\tilde\beta\rangle$ for $\tilde\alpha,\tilde\beta\in R^a$.

Definition 2.1. *a) A set $G=\{G_{\tilde\alpha}\in\mathfrak{F},\tilde\alpha\in R^a_+\}$ is an R-matrix if*

$$G_{\tilde\alpha}G_{\tilde\beta}=G_{\tilde\beta}G_{\tilde\alpha}, \tag{2.1}$$

$$G_{\tilde\alpha}G_{\tilde\alpha+\tilde\beta}G_{\tilde\beta}=G_{\tilde\beta}G_{\tilde\alpha+\tilde\beta}G_{\tilde\alpha}, \tag{2.2}$$

$$G_{\tilde\alpha}G_{\tilde\alpha+\tilde\beta}G_{\tilde\alpha+2\tilde\beta}G_{\tilde\beta}=G_{\tilde\beta}G_{\tilde\alpha+2\tilde\beta}G_{\tilde\alpha+\tilde\beta}G_{\tilde\alpha}, \tag{2.3}$$

$$G_{\tilde\alpha}G_{3\tilde\alpha+\tilde\beta}G_{2\tilde\alpha+\tilde\beta}G_{3\tilde\alpha+2\tilde\beta}G_{\tilde\alpha+\tilde\beta}G_{\tilde\beta}=G_{\tilde\beta}G_{\tilde\alpha+\tilde\beta}G_{3\tilde\alpha+2\tilde\beta}G_{2\tilde\alpha+\tilde\beta}G_{3\tilde\alpha+\tilde\beta}G_{\tilde\alpha} \tag{2.4}$$

under the assumption that $\tilde\alpha,\tilde\beta\in R^a_+$ and

$$\mathbf{R}\langle\tilde\alpha,\tilde\beta\rangle\cap R^a=\{\pm\gamma\},\gamma\ \text{ runs over all the indices} \tag{2.5}$$

in the corresponding identity.

b) A non-affine R-matrix $\{G_\alpha\in\mathfrak{F},\alpha\in R_+\}$ has to obey the same relations for $\alpha,\beta\in R_+$. A closed R-matrix (or a closure of the above G) is a set $\{G_{\tilde\alpha}\in\mathfrak{F},\tilde\alpha\in R^a\}$ (extending G and) satisfying relations (2.1 - 2.4) for arbitrary (maybe negative) $\tilde\alpha,\tilde\beta\in R^a$ such that the corresponding condition (2.5) is fulfilled. Non-affine closed R-matrices are defined for $\alpha\in R$.

□

The condition (2.5) for identity (2.1) means that

$$(\tilde\alpha,\tilde\beta)=0\ \text{ and }\ \mathbf{R}\langle\tilde\alpha,\tilde\beta\rangle\cap R^a=\{\pm\tilde\alpha,\pm\tilde\beta\}, \tag{2.6}$$

i.e. there exists $\tilde w\in W^a$ such that $\tilde\alpha=\tilde w(\alpha_i),\tilde\beta=\tilde w(\alpha_j)$ for simple $\alpha_i\neq\alpha_j(0\le i,j\le n)$ disconnected in Γ^a. The corresponding assumptions for (2.2-2.4) give that $\tilde\alpha,\tilde\beta$ are simple roots of a certain two-dimensional root subsystem in R^a (or R) of type A_2,B_2,G_2. Here $\tilde\alpha,\tilde\beta$ stay for α_1,α_2 in the notations from the figure of the systems of rank 2 from [B] . One can represent them as follows: $\tilde\alpha=\tilde w(\alpha_i),\tilde\beta=\tilde w(\alpha_j)$ for a proper $\tilde w$ from W^a and joined (neighbouring) α_i,α_j.

We will use the following formal notations:

$$^{\hat w}(G_{\tilde\alpha})=G_{\hat w(\tilde\alpha)},\quad ^{\hat w}(G_{\tilde\alpha}G_{\tilde\beta})=G_{\hat w(\tilde\alpha)}G_{\hat w(\tilde\beta)},..., \tag{2.7}$$

where $\hat w$ is from W^b, the roots $\tilde\alpha,\tilde\beta$ are from R^a. We do not assume here that W^b acts on $\mathfrak{F}$. However one can consider $\{G_{\tilde\alpha}\}$ as formal symbols satisfying the relations of the Definition 2.1 and $\mathfrak{F}$ as the free algebra generated by them over $\mathbf{C}$. Then indeed the above formulas define an action.

Proposition 2.2. *If G is an affine R-matrix then there exists a unique set $\{G_{\hat w},\hat w\in W^b\}$ satisfying the (homogeneous 1-cocycle) relations*

$$G_{xy}={}^{y^{-1}}G_xG_y,\quad G_{s_j}=G_j\overset{def}{=}G_{\alpha_j},G_{id}=1, \tag{2.8}$$

where $0 \le j \le n,\ x, y \in W^b$ *and* $l(xy) = l(x) + l(y)$.

□

Given a reduced decomposition $\hat{w} = \pi_r s_{j_l} \cdots s_{j_1},\ l = l(\hat{w}), r \in O$, one has (see (1.8)):

$$G_{\hat{w}} = \hat{w} G_{\tilde{\alpha}^l} \cdots G_{\tilde{\alpha}^1},\ \tilde{\alpha}^1 = \alpha_{j_1}, \tilde{\alpha}^2 = s_{j_1}(\alpha_{j_2}), \tilde{\alpha}^3 = s_{j_1} s_{j_2}(\alpha_{j_3}), \dots . \tag{2.9}$$

We always have the following closures (*the unitary one* defined for invertible G and *the extension by 0*):

$$G_{-\tilde{\alpha}} = G_{\tilde{\alpha}}^{-1}, \quad G_{-\tilde{\alpha}} = 0, \quad \tilde{\alpha} \in R_+^a. \tag{2.10}$$

If there exists an action of $W^a \ni \tilde{w}$ on $\mathfrak{F}$ such that

$$\tilde{w}(G_{\tilde{\alpha}}) = {}^{\tilde{w}}G_{\tilde{\alpha}} = G_{\tilde{w}(\tilde{\alpha})} \text{ for } \tilde{\alpha},\ \tilde{w}(\tilde{\alpha}) \in R_+^a,$$

then the extension of G satisfying these relations for all $\tilde{w}$ is well-defined and closed (*the invariant closure*).

Proposition 2.3. *Let us suppose that the non-affine R-matrix G is closed and*

$$G_\alpha G_\beta = G_\beta G_\alpha \quad \textit{for long roots such that } (\alpha, \beta) = 0. \tag{2.11}$$

In the case of G_2 we take all orthogonal roots in (2.11) and add condition (2.2) for long α, β. We call such an R-matrix extensible. Assuming that the group $B \ni a$ operates on the algebra $\mathfrak{F} \ni f$ (written $f \to b(f)$) and

$$b(G_\alpha) = G_\alpha \quad \textit{whenever} \quad (b, \alpha) = 0,\ b \in B, \alpha \in R, \tag{2.12}$$

we set

$${}^{b'}G_\alpha = G_{\tilde{\alpha}} \stackrel{def}{=} b(G_\alpha), \quad \textit{if } \tilde{\alpha} = b'(\alpha) = [\alpha, -(b, \alpha)] \tag{2.13}$$

for arbitrary $\alpha \in R, b \in B$. Then $G_{\tilde{\alpha}}$ are well-defined (depend on the corresponding scalar products (b, α) only) and form a closed affine R-matrix.

□

From now on we assume that affine G is obtained by this construction. We see that $G_{\hat{w}}$ is the product of $G_{\tilde{\alpha}}$ when $\tilde{\alpha}$ runs over $\lambda(\hat{w})$. It does not depend on the choice of the closure of G if λ_w contains no elements $\tilde{\alpha} = [\alpha, k]$ with $\alpha < 0$ (we call such $w \in W^b$ *dominant*). Moreover, the elements G_w for non-dominant w vanish if the closure is the extension by 0. If the closure is unitary, then formulas (2.12) are valid for any $x, y \in W^b$.

Proposition 2.4. *The elements from $B'_+ = \oplus_{i=1}^n \mathbf{Z} b'_i$ are dominant. Given $b, c \in B_+,\ l((b+c)') = l(b') + l(c')$, and*

$$G_{b'+c'} = (-c)(G_{b'})\, G_{c'} = (-b)(G_{c'})\, G_{b'}, \tag{2.14}$$

$$G_{b'_r} = G_{\omega_r}, \quad \textit{where } r \in O^*,\ G_{\theta'} = G_{[\theta,1]} G_{s_\theta}. \tag{2.15}$$

□

DEFINITION 2.5. *The quantum Knizhnik- Zamolodchikov equation is one of the following equivalent systems of relations for an element* $\Phi \in \mathfrak{F}$ *:*

$$
\begin{aligned}
&i)\ b_i^{-1}\ (\Phi)\ =\ G_{b_i'}\ \Phi\ ,\ 1 \le i \le n\ ;\\
&ii)\ b^{-1}\ (\Phi)\ =\ G_{b'}\ \Phi\ \ \textit{for}\ \ \ \textit{any}\ b \in B_+.
\end{aligned}
\tag{2.16}
$$

□

Let us assume that there is an action of W^b on $\mathfrak{F}$ making G invariant or/and Φ is a series (or any expression) in terms of $\{G_{\tilde\alpha}\}$. For invertible Φ, the (homogeneous) *monodromy cocycle* $\{\mathcal{G}_w, w \in W\}$ is determined by the relations:

$$
\begin{aligned}
&\mathcal{G}_i = \mathcal{G}_{s_i} \stackrel{def}{=} {}^{w^{-1}}(\Phi^{-1}) G_i \Phi \ \text{ for } \ 1 \le i \le n,\ \mathcal{G}_{id} = 1,\\
&\mathcal{G}_{xy} = {}^{y^{-1}}\mathcal{G}_x \mathcal{G}_y \ \text{ whenever } \ l(xy) = l(x) + l(y).
\end{aligned}
\tag{2.17}
$$

The existence results from the relations for $\{\mathcal{G}_i\}$ from Definition 2.1.

Let us suppose that the limits

$$
\Phi\ =\ \lim_{b\to+\infty} b(G_{b'}),\ {}^{w}\Phi\ =\ \lim_{b\to+\infty} w(b)({}^{w}G_{b'}),\ w \in W,
\tag{2.18}
$$

are well-defined in the algebra $\mathfrak{F}$ together with their finite products as all $\{k_i\}$ from the decomposition $b = \sum_{i=1}^n k_i b_i \in B_+$ tend to ∞. Then Φ is a solution of (2.16). We assume that the action of B is continuous in the topology of $\mathfrak{F}$.

PROPOSITION 2.6. *The following products*

$$
\mathcal{G}_\alpha \stackrel{def}{=} \cdots G^{-1}_{[-\alpha,-2]} G^{-1}_{[-\alpha,-1]} G_\alpha G_{[\alpha,-1]} G_{[\alpha,-2]} \cdots
\tag{2.19}
$$

are convergent and form a non-affine closed extensible R-matrix $(\alpha \in R)$. *When* $\alpha = \alpha_i$ *they coincide with* $\mathcal{G}_i$ *from (2.17) for the above* Φ.

□

THEOREM 2.7. *In the above setup, assume that there is another continuous action of the group* B *(written* $b^\iota(\ )$*) on the algebra* $\mathfrak{F}$ *such that*

$$
\begin{aligned}
b^\iota(G_\alpha) = G_\alpha \quad &\textit{whenever} \quad (b,\alpha) = 0,\ b \in B, \alpha \in R,\\
b^\iota a'\ =\ a' b^\iota \quad &\textit{whenever} \quad (b,a) = 0,\ a, b \in B.
\end{aligned}
\tag{2.20}
$$

Then

$$
\mathcal{G}^\iota_{\tilde\alpha} \stackrel{def}{=} b^\iota(\mathcal{G}_\alpha), \quad \textit{if} \ \ \tilde\alpha = b'(\alpha) = [\alpha, -(b,\alpha)]
\tag{2.21}
$$

for $\alpha \in R, b \in B$ *are well-defined (depend on* (b,α) *only) and form a closed affine R-matrix.*

□

Examples will be considered in Section 4.

3. Double affine Hecke algebras

We denote the least common order of the elements of Π by m ($m = 2$ for D_{2k}, otherwise $m = |\Pi|$). Later on $\mathbf{C}_{\delta,q}$ means the field of rational functions in parameters $\{\delta^{1/m}, q_\nu^{1/2}\}, \nu \in \nu_R$. We will not distingwish b and b' in this and the next section. Let

$$q_{\tilde\alpha} = q_{\nu(\tilde\alpha)},\ q_j = q_{\alpha_j}, \quad \text{where}\ \ \tilde\alpha \in R^a, 0 \le j \le n. \tag{3.1}$$

Setting

$$x_i = \exp(b_i),\ x_b = \exp(b) = \prod_{i=1}^n x_i^{k_i} \ \text{ for }\ b = \sum_{i=1}^n k_i b_i,$$

$\mathbf{C}_\delta[x] = \mathbf{C}_\delta[x_b]$ means the algebra of polynomials in terms of $x_i^{\pm 1}$ with the coefficients depending on $\delta^{1/m}$ rationally. We will also use

$$X_{\tilde b} = \prod_{i=1}^n X_i^{k_i}\delta^k \ \text{ if }\ \tilde b = [b,k],\ b = \sum_{i=1}^n k_i b_i \in B,\ k \in \frac{1}{m}\mathbf{Z}, \tag{3.2}$$

where $\{X_i\}$ are independent variables which act in $\mathbf{C}_\delta[x]$ naturally:

$$X_{\tilde b}(p(x)) \ = \ x_{\tilde b}p(x), \ \ \text{where}\ \ x_{\tilde b} \stackrel{def}{=} x_b\delta^k,\ p(x) \in \mathbf{C}_\delta[x]. \tag{3.3}$$

The elements $\hat w \in W^b$ act in $\mathbf{C}_\delta[x],\ \mathbf{C}_\delta[X] = \mathbf{C}_\delta[X_b]$ by the formulas:

$$\hat w(x_{\tilde b}) \ = \ x_{\hat w(\tilde b)},\ \ \hat w X_{\tilde b}\hat w^{-1} \ = \ X_{\hat w(\tilde b)}. \tag{3.4}$$

In particular:

$$\pi_r(x_b) \ = \ x_{\omega_r^{-1}(b)}\delta^{(b_{r^*},b)} \ \text{ for }\ \alpha_{r^*} \stackrel{def}{=} \pi_r^{-1}(\alpha_0),\ r \in O^*. \tag{3.5}$$

We set $([a,k],[b,l]) = (a,b)$ for $a,b \in B,\ [\alpha,k]^\vee = [\alpha^\vee,k]$, and $a_0 = \alpha_0$.

Definition 3.1. *The double affine Hecke algebra* $\mathfrak{H}$ *(see [C1,C2]) is generated over the field* $\mathbf{C}_{\delta,q}$ *by the elements* $\{T_j,\ 0 \le j \le n\}$*, pairwise commutative* $\{X_b,\ b \in B\}$ *satisfying (3.2), and the group* Π *where the following relations are imposed:*

(*o*) $(T_j + q_j^{1/2})(T_j - q_j^{-1/2}) \ = \ 0,\ 0 \ \le \ j \ \le \ n;$
(*i*) $T_iT_jT_i... \ = \ T_jT_iT_j...,$ m_{ij} *factors on each side;*
(*ii*) $\pi_r T_i \pi_r^{-1} \ = \ T_j$ *if* $\pi_r(\alpha_i) = \alpha_j;$
(*iii*) $T_i^{-1}X_bT_i^{-1} \ = \ X_bX_{a_i}^{-1}$ *if* $(b,\alpha_i) = 1,\ 1 \le i \le n;$
(*iv*) $T_0^{-1}X_bT_0^{-1} \ = \ X_{s_0(b)} \ = \ X_bX_\theta\delta^{-1}$ *if* $(b,\theta) = -1;$
(*v*) $T_iX_b \ = \ X_bT_i$ *if* $(b,\alpha_i) = 0$ *for* $0 \le i \le n;$
(*vi*) $\pi_rX_b\pi_r^{-1} \ = \ X_{\pi_r(b)} \ = \ X_{\omega_r^{-1}(b)}\delta^{(b_{r^*},b)},\ r \in O^*.$

$\square$

Given $\tilde w \in W^a, r\in O,$ the product

$$T_{\pi_r\tilde w} \stackrel{def}{=} \pi_r \prod_{k=1}^{l} T_{i_k}, \quad \text{where} \quad \tilde w = \prod_{k=1}^{l} s_{i_k}, l = l(\tilde w), \tag{3.6}$$

does not depend on the choice of the reduced decomposition (because $\{T\}$ satisfy the same "braid" relations as $\{s\}$ do). Moreover,

$$T_{\hat v}T_{\hat w} \ = \ T_{\hat v\hat w} \ \text{ whenever } l(\hat v\hat w) = l(\hat v) + l(\hat w) \ \text{ for } \ \hat v, \hat w \in W^b. \tag{3.7}$$

In particular, we arrive at the pairwise commutative elements (use (3.7) and Proposition 2.4):

$$Y_b \ = \ \prod_{i=1}^{n} Y_i^{k_i} \ \text{ if } \ b = \sum_{i=1}^{n} k_i b_i \in B, \quad \text{where} \ \ Y_i \stackrel{def}{=} T_{b_i}. \tag{3.8}$$

PROPOSITION 3.2.

$$\begin{aligned} T_i^{-1}Y_bT_i^{-1} \ &= \ Y_bY_{a_i}^{-1} \quad \textit{if} \ \ (b,\alpha_i) = 1, \\ T_iY_b \ &= \ Y_bT_i \quad \textit{if} \ \ (b,\alpha_i) = 0, \ 1\le i\le n. \end{aligned} \tag{3.9}$$

□

We note that the above relations are equivalent to the following:

$$T_i^{-1}Y_iT_i^{-1} = Y_iY_{a_i}^{-1}, \ T_iY_j = Y_jT_i \ \text{ for } \ 1\le i\ne j\le n. \tag{3.10}$$

PROPOSITION 3.3. *The following three maps define automorphisms of* $\mathfrak{H}$*:*

$$\begin{aligned} \varepsilon : X_i \to Y_i^{-1}, \quad & Y_i \to X_i^{-1}, \quad T_i \to T_i^{-1}, \\ & q_\nu \to q_\nu^{-1}, \ \delta \to \delta^{-1}, \ 1\le i\le n, \ \nu\in\nu_R, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \tau : X_i \to X_i, \quad & Y_i \to X_iY_i\delta^{-c_i}, \quad T_i \to T_i, \\ & q_\nu \to q_\nu, \ \delta\to\delta, \ c_i = (b_i,\rho_2)/(1+(\theta,\rho_2)), \\ \omega : X_i \to Y_i, \quad & Y_i \to Y_i^{-1}X_iY_i\delta^{2c_i}, \quad T_i\to T_i, \\ & q_\nu\to q_\nu, \ \delta\to\delta, \ 1\le i\le n, \ \nu\in\nu_R. \end{aligned} \tag{3.12}$$

□

This theorem can be either deduced from the topological interpretation of the double braid group from [C1] (defined by relations (i)-(vi), Definition 3.1) or checked by direct consideration. One has to add proper roots of δ to $\mathfrak{H}$ when defining τ,ω (corresponding to the standard generators of $SL_2(\mathbf{Z})$ in the case $q = 1 = \delta$). The first automorphism can be interpreted as the self-duality of the (formal) zonal Fourier transform in the difference case (see [DJO] for the differential rational case). Conjecturally the symmetric polynomials of

the images of $\{Y_1, \ldots, Y_n\}$ with respect to the action of the proper products of τ, ω generate certain difference analogs of the so-called W_∞-algebras from conformal field theory.

The *Demazure-Lusztig operators* (see [KL, KK, C1], and [C5] for more detail)

$$(3.13)\quad \hat{T}_j \ = \ q_j^{-1/2} s_j \ + \ (q_j^{-1/2} - q_j^{1/2})(X_{a_j}^{-1} - 1)^{-1}(s_j - 1), \ 0 \le j \le n.$$

act in $\mathbf{C}_{\delta,q}[x]$ naturally. We note that only $\hat{T}_0$ depends on δ:

$$(3.14)\quad \begin{aligned} \hat{T}_0 \ &= q_0^{-1/2} s_0 \ + \ (q_0^{-1/2} - q_0^{1/2})(\delta^{-1} X_\theta - 1)^{-1}(s_0 - 1), \\ \text{where} \ \ s_0(X_i) \ &= \ X_i X_\theta^{-(b_i,\theta)} \delta^{(b_i,\theta)}. \end{aligned}$$

THEOREM 3.4. *i) The map $T_j \to \hat{T}_j$, $X_b \to X_b$ (see (3.2,3.3)), $\pi_r \to \pi_r$ (see (3.5)) induces a $\mathbf{C}_{\delta,q}$-linear homomorphism from $\mathfrak{H}$ to the algebra of linear endomorphisms of $\mathbf{C}_{\delta,q}[x]$.*

ii) This representation is faithful and remains faithful when δ, q take any non-zero values assuming that δ is not a root of unity. Elements $H \in \mathfrak{H}$ and their images $\hat{H}$ have the unique decompositions

$$(3.15)\quad H = \sum_{b\in B, w\in W} Y_b h_{b,w} T_w, \quad h_{b,w} \in \mathbf{C}_{\delta,q}[X].$$

$$(3.16)\quad \hat{H} \ = \sum_{b\in B, w\in W} b g_{b,w} w \ = \sum_{b\in B, w\in W} b(g_{b,w}) \ bw,$$

where $g_{b,w}$ belong to the field $\mathbf{C}_{\delta,q}(X)$ of rational functions in $\{X_1, ..., X_n\}$. □

PROPOSITION 3.5. *i) Given $b \in B$ and $\hat{w} = \pi_b \omega$, $\omega \in W$,*

$$(3.17)\quad \hat{T}_{\hat{w}} = <\hat{T}_{\hat{w}}> + \sum_{b_*, w\in W} b_* g_{b_*,w} w,$$

summed over $b_* \in \sigma^\vee(b)$ such that $\ell(b_*) < \ell(b)$, where $g_{b,w} \in \mathbf{C}_{\delta,q}(X)$,

$$(3.18)\quad <\hat{T}_{\hat{w}}> = \prod_{\tilde{\alpha}\in\lambda(\pi_b^{-1})} \frac{q_{\tilde{\alpha}}^{1/2} X_{\tilde{\alpha}^\vee} - q_{\tilde{\alpha}}^{-1/2}}{X_{\tilde{\alpha}^\vee} - 1} b\omega_b^{-1} \hat{T}_\omega.$$

ii) If $b \in B_+$, then $\pi_{-b} = -b$, $Y_{-b} = T_b^{-1}$, and

$$(3.19)\quad \begin{aligned} \hat{Y}_{-b} &= <\hat{Y}_{-b}> + \sum_{b_*, w\in W} b_* g_{b_*,w} w, \qquad b \neq b_* \in \sigma^\vee(b), \\ <\hat{Y}_{-b}> &= \prod_{\tilde{\alpha}\in\lambda(b)} \frac{q_{\tilde{\alpha}}^{1/2} X_{\tilde{\alpha}^\vee} - q_{\tilde{\alpha}}^{-1/2}}{X_{\tilde{\alpha}^\vee} - 1}(-b). \end{aligned}$$

□

4. Difference-elliptic operators

The above considerations lead to the following affine R-matrix:

$$G_{\tilde\alpha;q} = G_{\tilde\alpha} = 1 + (q_{\tilde\alpha} - 1)(X_{\tilde\alpha^\vee} - 1)^{-1}(s_{\tilde\alpha} - 1) = G^{-1}_{\tilde\alpha;q^{-1}},\ \tilde\alpha \in R^a. \tag{4.1}$$

Given a reduced decomposition $\hat w = \pi_r s_{j_l} \cdots s_{j_1},\ l = l(\hat w), r \in O$, one has (see (3.6)):

$$\begin{aligned} \hat T_{\hat w} \ &= \ \hat w \prod_\nu q_\nu^{-l_\nu(\hat w)/2} G_{\tilde\alpha^l} \cdots G_{\tilde\alpha^1}, \\ \tilde\alpha^1 &= \alpha_{j_1}, \tilde\alpha^2 = s_{j_1}(\alpha_{j_2}), \tilde\alpha^3 = s_{j_1}s_{j_2}(\alpha_{j_3}), \ldots . \end{aligned} \tag{4.2}$$

We will apply the procedure of Proposition 2.6 to the G. From now on we fix $q \neq 0$.

Definition 4.1. *Given $æ > 0,\ M > 1$, let*

$$\begin{aligned} \Xi_{æ}(M) \ = \ \{\ (x,\delta) \ &\text{such that} \ \ |\delta| \le \exp(-æ), \\ &|(x_\alpha \delta^k - 1)^{-1}| < M > |x_\alpha|\ \}, \end{aligned} \tag{4.3}$$

for all $k \in \mathbf{Z},\ \alpha \in R$, where $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$. We denote by $\mathfrak{F}$ the algebra of the series

$$f = \sum_{\hat w} f_{\hat w}(x,\delta)\hat w \ \ \textit{for} \quad \textit{scalar } f_{\hat w},\ \hat w \in W^b, \tag{4.4}$$

satisfying the following condition. There exists a constant $c_f > 0$ (depending on f) such that for any $M > 1, æ$, and $æ \ge \epsilon > 0$ the functions $\{f_{\hat w}\}$ are continuous in $\Xi_{æ}(M)$ and the series (4.4) is convergent with respect to the norm

$$\begin{aligned} ||f|| &\stackrel{def}{=} \sum_{\hat w} \sup\{|f_{\hat w}| \ \textit{in}\ \Xi_{æ}(M)\} ||\hat w||, \\ \textit{where} \ \ ||\hat w|| \ &= \ \exp(c_f(æ - \epsilon) l(\hat w)). \end{aligned} \tag{4.5}$$

Here $l(\hat w)$ is the length of $\hat w \in W^b$, $|\ |$ the absolute value.

□

One can follow [C3],Proposition 2.5 to check that $\mathfrak{F}$ is really an algebra.

Let us add two more variables ξ, ζ and one more parameter t to $\mathbf{C}_\delta[x]$ and extend the action of $b \in B$ and W^b setting

$$b(\xi) = \xi x_b \delta^{-(b,b)/2},\ w(\xi) = \xi,\ w \in W,\ b(\zeta) = \zeta. \tag{4.6}$$

Replacing δ by t in formulas (3.2-3.5) and ξ by ζ, we introduce another action of $b \in B$:

$$b^\iota(\zeta) = \zeta x_b t^{-(b,b)/2},\ w(\zeta) = \zeta,\ w \in W,\ b^\iota(\xi) = \xi. \tag{4.7}$$

The group $\mathcal{W}$ of automorphisms of $\mathbf{C}_{\delta,t}[x,\xi,\zeta]$ generated by W^b, B^ι is isomorphic to the following central extension of the the semi-direct product $W{\ltimes}(B\oplus B^\iota)$ by $\mathbf{Z}=\{d^l, l\in\mathbf{Z}\}$. The action of W on B, B^ι by conjugation is given by the same formulas as in Section 1. However

$$
\begin{aligned}
a^\iota b \ &= \ ba^\iota d^{(a,b)} \ \text{ for } \ a,b\in B, \ \text{ where}\\
d(x_b) = x_b, \ d(\xi) \ &\stackrel{def}{=} \xi t^{-1}, \ d(\zeta) \ \stackrel{def}{=} \zeta\delta.
\end{aligned}
\tag{4.8}
$$

Formulas (4.6,4.7) will not be applied in this paper but the last one will. We note that the action of $\mathcal{W}$ in $C[x]$ remains faithful when δ, t are considered as numbers assuming that $t^r\delta^s=1$ iff $r=0=s$ for integral r,s. We will call $\mathcal{W}$ *2-extended Weyl group* due to K. Saito.

Thanks to (4.8) we can construct the element Φ from (2.18) and the R-matrix $\mathcal{G}$ (Proposition 2.6) which belong to $\mathfrak{F}$. Let us extend it to the affine closed W^b_ι-invariant R-matrix $\{\mathcal{G}^\iota_{\tilde\alpha}\}$, where the action is by conjugation relative to (4.7) (see Theorem 2.7). This R-matrix belongs to the algebra $\mathfrak{F}^\#$ of finite sums $\sum_{\hat w,r} F_{\hat w,r}(x,\delta)\hat w^\iota d^r$, where $F_{\hat w}$ for $\hat w\in W^b$, $r\in(1/m)\mathbf{Z}$ are finite products of series (4.4) "shifted" by any b^ι.

Introducing $\mathcal{G}^\iota_{\hat w}$ for $\hat w\in W^b$, we set

$$
\begin{aligned}
\mathcal{T}_{\hat w} \ &= \ \prod_\nu q_\nu^{-l_\nu(\hat w)/2}\hat w\mathcal{G}^\iota_{\hat w} \ \text{ for } \ \hat w\in W^b,\\
\mathcal{Y}_b \ &= \ \mathcal{T}_b, \ \mathcal{Y}_{a-b} \ = \ \mathcal{Y}_a\mathcal{Y}_b^{-1}, \ \text{ for } \ a,b\in B_+.
\end{aligned}
\tag{4.9}
$$

THEOREM 4.2. *The map ϕ sending*

$$
\begin{aligned}
&T_i\to\mathcal{T}_i \stackrel{def}{=} \Phi^{-1}T_i\Phi \ \textit{ for } \ 1\le i\le n,\\
&Y_i\to\mathcal{Y}_i \stackrel{def}{=} \prod_\nu q_\nu^{-(\rho_\nu,b_i)}b_i^\iota\mathcal{G}_i\\
&X_i\to\mathcal{X}_i \stackrel{def}{=} \Phi^{-1}(Y_i)\Phi = \prod_\nu q_\nu^{-(\rho_\nu,b_i)}b_i,\\
&\delta\to d^{-1}\Delta, \ \Delta \stackrel{def}{=} q_0^{-1}\prod_\nu q_\nu^{-(\rho_\nu,\theta)}
\end{aligned}
\tag{4.10}
$$

can be extended to a homomorphism of $\mathfrak{H}$.

□

Let $p(x_1,\ldots,x_n)$ belong to the algebra $\mathbf{C}[x]^W$ of W-invariant elements in the $\mathbf{C}[x]$. We set

$$
\begin{aligned}
&\mathcal{L}_p = p(\mathcal{Y}_1,\ldots,\mathcal{Y}_n) = \sum_{b\in B,\hat w\in W^b,r} f_{b,\hat w,r}b^\iota\hat w d^r, \ r\in\frac{1}{m}\mathbf{Z},\\
&(\mathcal{L}_p)_{red} = \sum_{b,\hat w,r} f_{b,\hat w,r}b^\iota d^r, \ L_p \stackrel{def}{=} \sum_{b,\hat w,r} f_{b,\hat w,r}b^\iota\Delta^r.
\end{aligned}
\tag{4.11}
$$

Theorem 4.3. *The operators* $\{L_p, \;\; for \;\; p \in \mathbf{C}[y]^W\}$ *are pairwise commutative,* W^b*-invariant (i.e* $\hat{w}L_p\hat{w}^{-1} = L_p$ *for all* $\hat{w} \in W^b$*). Given p, there exists a finite set* $\Lambda \subset \mathbf{Z}$ *such that the coefficients* $g_b = \sum_{\hat{w}\in W^b,r} f_{b,\hat{w},r}\Delta^r$ *from the decomposition* $L_p = \sum_{b\in B} g_b b^\iota$ *are absolutely convergent series in*

$$\begin{aligned} \Xi_{æ,t}(M;\Lambda) \; = \; \{ \; (x,\delta,t), \; t \neq 0, \; |\delta| \leq \exp(-æ), \\ |(x_\alpha \delta^k t^l - 1)^{-1}| < M > |x_\alpha|, \quad where \;\; k \in \mathbf{Z}, \; l \in \Lambda \; \} \end{aligned} \tag{4.12}$$

for any $M > 1, æ > 0$*. They satisfy the following periodicity condition:*

$$a(g_b(x)) \; = \; \Delta^{(a,b)} g_b(x) \;\; for \;\; a, b \in B.$$

An outline of the proof. First, the operators $\mathcal{L}_p$ are pairwise commutative. Then $\mathbf{C}[X]^W$ belongs to the center of $\mathfrak{H}$ when $\delta = 1$. which follows from Definition 3.1 and Theorem 3.4. The same holds true for $\mathbf{C}[Y]^W$ because of Proposition 3.3 (use ε). Hence (see Theorem 4.2) the above operators belong to the center of the image of $\mathfrak{H}$ relative to ϕ. It gives the W^b-invariance which directly results in the statements of the theorem. □

We mention without going into detail that $\{L_p\}$ are formally self-adjoint with respect to the "elliptic" Macdonald pairing and preserve some remarkable finite dimensional subspaces (cf. [C3]). Let us demonstrate that the above construction gives also the commutativity of the operators from [R] (generalized to arbitrary root systems). We set

$$\begin{aligned} \mathcal{L}_p \; &= \sum_{a,b\in B, w\in W, r} f_{a,b,w,r} a b^\iota w d^r, \; r \in \frac{1}{m}\mathbf{Z}, \\ \mathcal{L}_p^0 &= \sum_{a,b,w,r} f_{a,b,w,r} a b^\iota d^r, \; L_p^0 \overset{def}{=} \sum_{b,w} f_{0,b,w,0}\, b^\iota. \end{aligned} \tag{4.13}$$

Then $\mathcal{L}_p^0 \mathcal{L}_g^0 = \mathcal{L}_{pg}^0$ for any W-invariant p, g.

Considering $p_r = \sum_{w\in W} w(b_r)$ for $r \in O^*$ and following Proposition 3.4 from [C2], we see that $\mathcal{L}_r^0 \overset{def}{=} \mathcal{L}_{p_r}^0 = \sum_{a,b,w} f_{a,b,w,0}\, a b^\iota$, where $(a, b) > 0$ for any non-zero pairs a, b. Hence

$$\begin{aligned} \mathcal{L}_s^0 \mathcal{L}_r^0 = \mathcal{L}_{p_s p_r}^0 = \mathcal{L}_s^0 (\sum_{a,b,w,r} f_{a,b,w,0}\, b^\iota a d^{-(a,b)}) \;\; \text{and} \\ L_s^0 L_r^0 = L_{p_s p_r}^0 = L_r^0 L_s^0 \;\; \text{for} \;\; L_r^0 = L_{p_r}^0, \; r, s \in O^*. \end{aligned} \tag{4.14}$$

The calculation of L_r^0 is not complicated (see Proposition 3.5) and results in

PROPOSITION 4.4. *The operators*

$$L_r^0 \;=\; \sum_{w\in W}\Big(\prod_{\alpha\in\lambda(b_r),k\geq 0}\frac{1-X_{w(\alpha^\vee)}q_\alpha\delta^k}{1-X_{w(\alpha^\vee)}\delta^k}$$

$$\prod_{\alpha\in\lambda(b_r),k>0}\frac{1-X^{-1}_{w(\alpha^\vee)}q_\alpha^{-1}\delta^k}{1-X^{-1}_{w(\alpha^\vee)}\delta^k}\Big)\,w(-b_r^\iota) \tag{4.15}$$

are pairwise commutative and W-invariant.

□

The last application will be the following parametric deformation of the above G and $\mathcal{G}$. Let us introduce the second system of the (same) groups $W'\subset(W^b)'\subset\mathcal{W}'$ and the corresponding $\{X',\delta',t'\}$, assuming that $\mathcal{W},\{X\}$ commute with $\mathcal{W}',\{X'\}$. We set

$$F_{\tilde\alpha}=\frac{G_{\tilde\alpha}+(1-q_{\tilde\alpha})(X'_{\tilde\alpha^\vee}-1)^{-1}s_{\tilde\alpha}}{1+(1-q_{\tilde\alpha})(X'_{\tilde\alpha^\vee}-1)^{-1}},\ \tilde\alpha\in R^a,$$

$$\mathcal{F}_\alpha\stackrel{def}{=}\cdots F_{[\alpha,2]}F_{[\alpha,1]}F_\alpha F_{[\alpha,-1]}F_{[\alpha,-2]}\cdots\ . \tag{4.16}$$

Then F is an invariant affine closed R-matrix with respect to the diagonal action of W^b ($\hat w(\hat w)'$ instead of $\hat w$). Moreover it is unitary ($F_{-\tilde\alpha}=F_{\tilde\alpha}^{-1}$). Hence $\{\mathcal{F}_\alpha\}$ is an invariant unitary R-matrix as well as its affine extension $\{\mathcal{F}_{\tilde\alpha}\}$. The latter is defined due to Theorem 2.7 relative to the diagonal action of $(W^b)^\iota$. The definition of the proper algebra $\mathfrak{F}$ containing $\mathcal{F}_{\tilde\alpha}$ is a direct version of Definition 4.1 for two sets of variables (see [C3], Proposition 2.5). Following Definition 2.5 we can use this $\mathcal{F}$ to introduce the *affine difference-elliptic quantum Knizhnik-Zamolodchikov equation*, which is connected with the eigenvalue problem for the above operators $\{L_p\}$ (cf. [C3], Theorem 3.5).

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REFERENCES

[B] N. BOURBAKI, *Groupes et algèbres de Lie*, Ch. **4–6**, Hermann, Paris (1969).

[C1] I.CHEREDNIK, Double affine Hecke algebras, Knizhnik- Zamolodchikov equations, and Macdonald's operators, IMRN (Duke M.J.) 9 (1992), 171–180.

[C2] ———, Double affine Hecke algebras and Macdonald's conjectures, Annals of Mathematics (1994).

[C3] ———, Elliptic quantum many- body problem and double affine Knizhnik- Zamolodchikov equation, Commun. Math. Phys. (1994).

[C4] ———, Quantum Knizhnik- Zamolodchikov equations and affine root systems, Commun. Math. Phys. 150 (1992), 109–136.

[C5] ———, Integration of quantum many- body problems by affine Knizhnik–Zamolodchikov equations, Preprint RIMS– 776 (1991), (Advances in Math.(1994)).

[DJO] C.F. Dunkl, M.F.E. de Jeu, and E.M. Opdam, Singular polynomials for finite reflection groups, Preprint (1994).

[FR] I. Frenkel, and N. Reshetikhin, Quantum affine algebras and holonomic difference equations, Commun. Math. Phys. 146 (1992), 1–60.

[He] G.J. Heckman, An elementary approach to the hypergeometric shift operators of Opdam, Invent.Math. 103 (1991), 341–350. omp. Math. 64 (1987), 329–352. v.Math. 70 (1988), 156–236.

[KL] D. Kazhdan, and G. Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras, Invent.Math. 87(1987), 153–215.

[KK] B. Kostant, and S. Kumar, T-Equivariant K-theory of generalized flag varieties, J. Diff. Geometry 32(1990), 549–603.

[M1] I.G. Macdonald, A new class of symmetric functions , Publ.I.R.M.A., Strasbourg, Actes 20-e Seminaire Lotharingen, (1988), 131–171 .

[M2] ———, Orthogonal polynomials associated with root systems,Preprint(1988).

[N] M. Noumi, A realization of Macdonald's symmetric polynomials om quantum homogeneous spaces, in : Proceedings of the 21st international conference on differential geometry methods in theoretical physics, Tian, China (1992) (to appear).

[O] E.M. Opdam, Some applications of hypergeometric shift operators, Invent.Math. 98 (1989), 1–18.

[R] S.N.M. Ruijsenaars, Complete integrability of relativistic Calogero-Moser systems and elliptic function identities, Communs Math. Phys. 110 (1987), 191–213.

[V] D-N.Verma, The role of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras, Lie groups and their representations (Proceedings of the Summer School on Group Representations), Budapest (1971), 653–705.